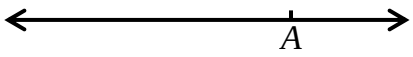
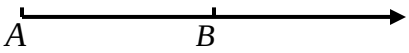
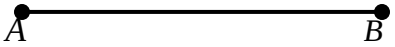
5.1 ANGLES AND ARCS

Definitions:

1. Half line 

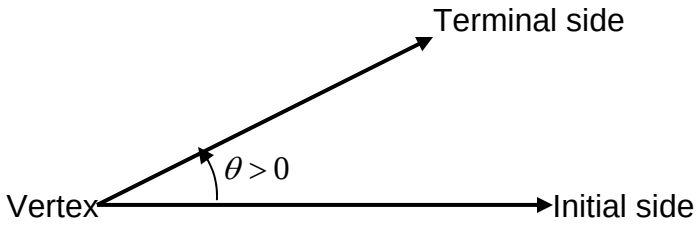
2. Ray 

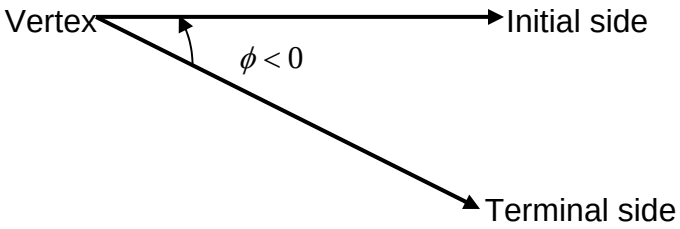
$\overrightarrow{AB}$

3. 

$\overline{AB}$

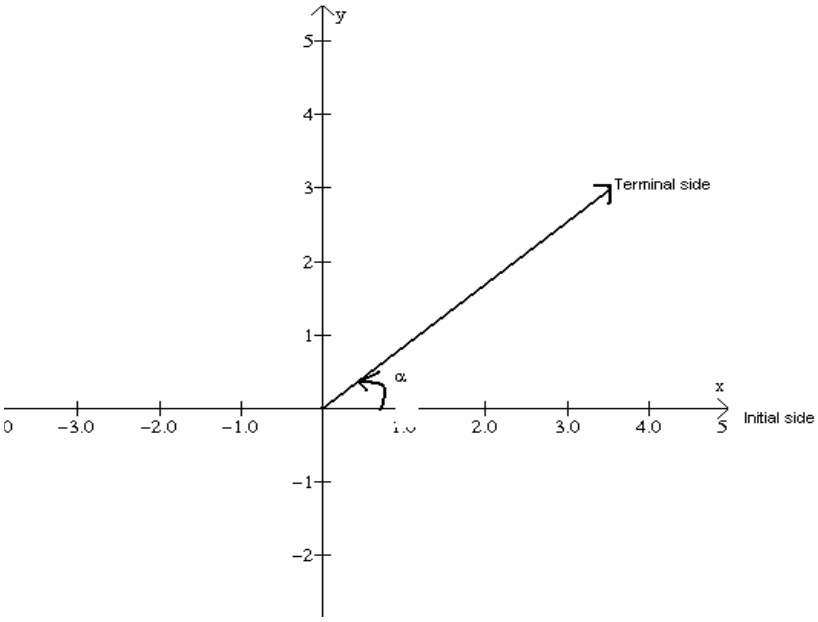
4. An angle consists of three components: vertex, initial side and the terminal side. An angle is a measure of how much turning has taken place about the vertex.





Positive angles are measured in counterclockwise direction
Negative angles are measured in clockwise direction.

An angle measured in ***standard position*** has its vertex at the origin and initial side on the positive x-axis as shown.



Units of measuring angles

$60''$ (secs)= $1'$ (min)

$60'$ (mins)= 1° (degree)

360° (degrees)= 1 revolution

$3600''$ (secs)= 1° (degree)

Fractions of a degree can be written as a decimal or in degree, minute, second (DMS).

Example 1

Express 22.125° in DMS

Solution

$22.125^\circ = 22^\circ + (0.125 \times 60)' = 22^\circ + 7.5'$

$\Rightarrow 22^\circ + 7' + (0.5 \times 60)''$

$\Rightarrow 22^\circ 7' 30''$

Example 2

Write the exact value of the following angle measure in decimal method

a) $48^\circ 30' 36''$

b) The difference between 65° and $29^\circ 14' 27''$

c) The sum of $41^\circ 40' 23''$ and $25^\circ 49' 55''$

Solution

a) $48^\circ 30' 36'' = 48^\circ + \left(\frac{30}{60}\right)^\circ + \left(\frac{36}{3600}\right)^\circ$

$\Rightarrow (48 + 0.5 + 0.01) = 48.51^\circ$

b)

$(65^\circ 0' 0'' - 29^\circ 14' 24'') = (64^\circ 59' 60'' - 29^\circ 14' 24'') = 35^\circ 45' 36''$

$\Rightarrow \left(35 + \frac{45}{60} + \frac{36}{3600}\right)^\circ = 35^\circ + 0.75^\circ + 0.01^\circ = 35.76^\circ$

c)

$(41^\circ 40' 23'' + 25^\circ 49' 55'') = 67^\circ 30' 18'' = \left(67 + \frac{30}{60} + \frac{18}{3600}\right)^\circ = 67^\circ + 0.5^\circ + 0.005^\circ = 67.505^\circ$

Types of Angles

The table below provides illustrates some types of angles that will used in this section.

Name	Angle	Comment
Right	90°	Terminal side on the positive y axis
Acute	$0^\circ < \theta < 90^\circ$	Terminal side in Quadrant 1
Obtuse	$90^\circ < \theta < 180^\circ$	Terminal side in Quadrant 2
Complimentary	$\phi + \theta = 90^\circ$	ϕ is the compliment of θ
Supplementary	$\phi + \theta = 180^\circ$	ϕ is the suppliment of θ
Quadrantal	$90^\circ.k$, $k \in \mathbb{Z}$	Terminal side is on the $x-$ or $y-$ axis
Co-terminal	$\theta + 360^\circ.k$, $k \in \mathbb{Z}$	Same Terminal side
Straight	180°	Terminal side on the negative $x-$ axis

Example 3

a) If the compliment of $29^\circ 16' 17''$ is α and the supplement of $32^\circ 3' 33''$ is β, write the exact value of $\alpha + \beta$ in DMS method

b) Find the positive angle less than 360° that is co-terminal with;

i) 555° ii) -1300°

Solution

a) $\alpha = \left(90^\circ 0' 0'' - 29^\circ 16' 17''\right) = 60^\circ 43' 43''$

$\beta = \left(180^\circ 0' 0'' - 32^\circ 2' 33''\right) = 147^\circ 56' 27''$

$\alpha + \beta = \left(60^\circ 43' 43'' + 147^\circ 56' 27''\right) = 208^\circ 40' 10''$

b) i) Smallest positive angle co-terminal with $555^\circ = 555^\circ + 360.k \quad k \in \mathbb{Z}$

$\Rightarrow 555^\circ - 360^\circ = 195^\circ$

ii) Smallest positive angle co-terminal with $-1300^\circ = -1300^\circ + 360.k \quad k \in \mathbb{Z}$

$\Rightarrow -1300^\circ + 1440^\circ = 140^\circ$

Classification of Angles in degrees

Angle in degrees	Quadrant
$0^\circ < \theta < 90^\circ$	1
$90^\circ < \theta < 180^\circ$	2
$180^\circ < \theta < 270^\circ$	3
$270^\circ < \theta < 360^\circ$	4

Angles that are co-terminal with each other belong to the same quadrant.

Example 4

Classify each of the following angles in standard position by the quadrant.

a) $495°$ b) $1115°$ c) $-727°$

Solution

a) Smallest positive angle coterminal with

$495^\circ = 495^\circ + 360^\circ.k = 495^\circ - 360^\circ = 135^\circ$

135° is quadrant 2 $\Rightarrow 495^\circ$ is quadrant 2

b) Smallest positive angle coterminal with

$1115^\circ = 1115^\circ + 360^\circ.k = 1115^\circ - 1080^\circ = 35^\circ$

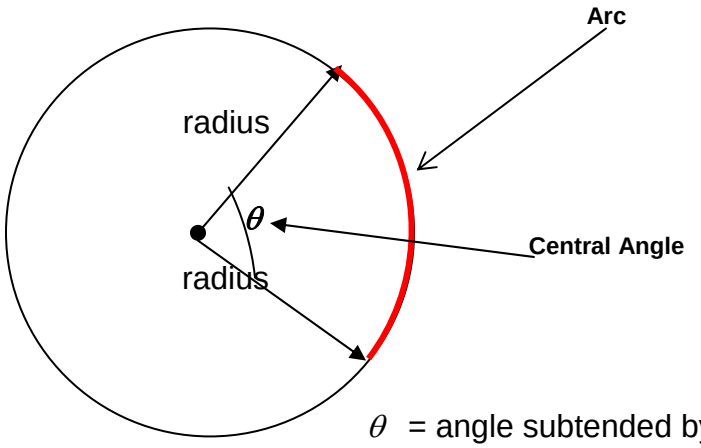
35° is quadrant 2 $\Rightarrow 1115^\circ$ is quadrant 1

c) Smallest positive angle coterminal with

$-727^\circ = -727^\circ + 360^\circ.k = -727^\circ + 1080^\circ = 353^\circ$

353° is quadrant 2 $\Rightarrow 353^\circ$ is quadrant 4

Measurement of Angles in Radians



θ = angle subtended by the Arc

Arc length = s = $2\pi r$

Angle in radians= $\theta = \frac{s}{r}$

If $s = 2\pi r$ is the arc length equivalent to one revolution

Then the angle in radians is $\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$ radians

1 revolution = $360^\circ \Leftrightarrow 2\pi$ Radians

Note: 1 radian = $\frac{180^\circ}{\pi}$ and $1^\circ = \frac{\pi}{180}$ radians

Example 5

Convert a) $270°$ to radians b) $\frac{3}{8}\pi$ radians to degrees

Solution

a) $270^\circ \Leftrightarrow \frac{270}{180} \times \pi = \frac{3\pi}{2}$ radians

b) $\frac{3}{8}\pi \Leftrightarrow \frac{3}{8} \times 180 = 67.5^\circ$

Example 6

Find the length of an arc that subtends an angle of 150° in a circle of diameter 16 cm.

Solution

$$S = r\theta = 8 \times \frac{150}{180} \times \pi = \frac{20\pi}{3} \text{ cm}$$

Angle Classification in Radians

Angle in Radians	Quadrant
$0 < \theta < \frac{\pi}{2}(1.57)$	1
$\frac{\pi}{2}(1.57) < \theta < \pi(3.14)$	2
$\pi(3.14) < \theta < \frac{3\pi}{2}(4.71)$	3
$\frac{3\pi}{2}(4.71) < \theta < 2\pi(6.28)$	4

Example 7

a) In which quadrant does 11 radians lie?

b) Find the **smallest positive** and the **largest negative angle** co-terminal with 13 radians

c) The difference between two complimentary angles is α and β is $\frac{\pi}{18}$ radians. Find the measure of angle α in terms of π.

Solution

a) The smallest positive angle co-terminal with 11 radians $= 11 - 2\pi \simeq 4.72$ which is quadrant 4. $\therefore$ 11 radians is in quadrant 4.

b) The smallest positive angle co-terminal with 13 radians $= 13 - 4\pi$

The largest negative angle co-co-terminal with 13 radians$= 13 - 6\pi$

c) $$\left.\begin{aligned} \alpha - \beta &= \frac{\pi}{18} \\ \alpha + \beta &= \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \alpha = \frac{5\pi}{18}$$

Linear and Angular Speed

Angular speed ω = Angle in radians ÷Time = $\frac{\theta}{t}$

Linear speed v = Distance along the Arc ÷ Time = $\frac{s}{t} = \frac{r\theta}{t} = r\omega$

Note: If v is m/s and r is in $meters$ then ω $rad/\sec$
If v is $cm/\min$ and r is in cm then ω $rad/\min$

Example 8

An air fan is rotating at 30 revolutions per minute. Find the angular speed in radians per second.

Solution

$$\omega = 30rev/\min = \frac{30\times 2\pi}{60} rad/\sec = \pi rad/\sec$$

Example 9
A wheel is moving at a linear speed of 30 meters per second. If the diameter of the wheel is 30 cm find the angular speed of the wheel in radians per minute.

Solution

Diameter = $= 2r \Rightarrow r = 15cm = \frac{15}{100} m$

$$\omega = \frac{v}{r} = \frac{30}{\frac{15}{100}} = \frac{30\times 100}{15} = 200rad/\sec$$

$$\Rightarrow 200\times 60 rad/\min = 12000 rad/\min$$